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Fluctuations in a Lévy Flight Gas

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We consider the density fluctuations of an ideal Brownian gas of particles performing Lévy flights characterized by the index f. We find that the fluctuations scale as $\Delta N(t) \sim t^H$, where the Hurst exponent H locks onto the universal value 1/4 for Lévy flights with a finite root-mean-square range ($f > 2$). For Lévy flights with a finite mean range but infinite root-mean-square range ($1 < f < 2$) the Hurst exponent $H = 1/(2f)$. For infinite-range Lévy flights ($f < 1$) the Hurst exponent locks onto the value 1/2. The corresponding power spectrum scales with an exponent $1 + 2H$, independent of dimension.

KEY WORDS: Lévy flight; Hurst exponent; Brownian motion; scaling argument; density correlations; temporal fluctuations; return times; simulation; power spectrum.

1. INTRODUCTION

It is of interest to investigate the influence of microscopic dynamical processes on the macroscopic properties of many-body systems. In particular, it is of great importance to investigate scaling behavior and the role of universality. Here we consider the macroscopic fluctuations of the density of a gas of independent and noninteracting particles. The individual particles are assumed to perform scale-invariant Lévy flights characterized by an exponent f.[1-6] This system is particularly simple since the random processes are additive and the system otherwise is entirely linear. We find that the temporal fluctuations of the density $\Delta N(t) \sim t^H$, independent of the spatial dimension. For $f > 2$ the root-mean-square range of the individual Lévy flight is finite, the central limit theorem[7] holds, and the Hurst exponent[2,6] H locks onto the universal value 1/4 characteristic of ordinary

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Brownian motion.[8,9] In the interval $1 < f < 2$ the root-mean-square range diverges, but the Lévy flight still has a range given by the mean value of the step size. In this case the Hurst exponent shows an anomalous behavior and depends explicitly on f, $H = 1/(2f)$, i.e., the macroscopic scale of the fluctuations depends on the microscopic Lévy distribution. For $0 < f < 1$ the Lévy flights have infinite range and H locks onto the value 1/2.

2. MODEL

We consider a system of particles in d-dimensional space. The particles perform independent isotropic random motion with a step size distribution given by the Lévy distribution,[3,7]

$$P(\mathbf{s})\, ds \sim s^{-1-f}\, ds \tag{1}$$

At short distances we introduce a microscopic cutoff representing the shortest length scale in the problem, i.e., a lattice distance or a molecular size. This UV cutoff is taken as our length unit. In order to ensure a proper normalization of $P(s)$, we must choose the characteristic exponent $f > 0$.

The macroscopic physics ensuing from the Lévy distribution for the microscopic elementary step depends entirely on the range characteristics of $P(s)$. For $f > 2$ the second moment $\langle s^2 \rangle \sim \int P(s)\, s^2\, ds$ exists and a characteristic step size is given by the root mean square deviation $\langle s^2 \rangle^{1/2}$. For $1 < f < 2$ the second moment diverges, but the mean range $\langle s \rangle$ is finite, defining an effective step size. In the interval $0 < f < 1$ the first moment diverges and a mean step size is not defined.

3. SCALING ARGUMENT

Before we make a more detailed analysis of the density fluctuation spectrum we present, in analogy with the analysis in ref. 8, a simple scaling argument for the temporal fluctuations of the particle density. The transition probability $P(rt \mid r't')$ for a Brownian particle carrying out a series of Lévy flights to progress from the point r at time t to the point r' at time t' is easily inferred, since only additive random processes are involved.[3,7]

$P(rt \mid r't')$ has the general scaling form[4]

$$P(rt \mid r't') \sim |t - t'|^{-d/\mu}\, F((r - r')/|t - t'|^{1/\mu}) \tag{2}$$

where the scaling exponent μ depends on the index f characterizing the

[4] We assume that the walker always uses one time unit to jump to the next position. This condition can be relaxed by introducing more general "Lévy walkers"[4,5] who traverse a given step with a prescribed velocity. If the first or second moment of the time step distribution becomes infinite, the scaling form in Eq. (2) will change. We shall not consider these general walkers further in this paper.

Lévy distribution. For $f>2$ the Lévy flights have a well-defined root mean square deviation, i.e., a finite second moment, and μ locks onto the value 2. For $f<2$ the Lévy flights trace out a fractal of dimension f,[2, 3, 6] and the scaling exponent is $\mu=f$. In the marginal case $f=2$ we find logarithmic corrections to the scaling form.[3, 10]

For $\mu=2$ the scaling function $F(x)$ takes the well-known Gaussian form $F(x)=\exp(-x^2)$; this is a consequence of the central limit theorem,[7] which in the present context leads to universal behavior. For $\mu<2$ the scaling function $F(x)$ can only be given explicitly in terms of known functions for $\mu=1$, where we find the Cauchy distribution $F(x)=(1+x^2)^{-(d+1)/2}$.[7] It is, however, easy to show that $F(x)\to\text{const}$ for $x\to 0$ and $F(x)\to 0$ for $x\to\infty$.

In order to estimate the particle number fluctuations $\Delta N(t)$, we consider a bounded volume of linear extent L, where L is much larger than the microscopic length scale in the problem, i.e., the mean displacement per unit time for $f>2$ or, otherwise, the small-distance cutoff for $f\leqslant 2$. The instantaneous number of particles in the volume $N(t)$ fluctuates about the mean value $N=\rho L^d$, where ρ is the mean particle density. To the extent that the elementary Lévy step has a well-defined size, i.e., for $1<\mu<2$, the scaling form in Eq. (2) defines an effective "dispersion law" for the propagation of a "Lévy flight" particle, $x\sim t^{1/\mu}$, hence the size L of the volume sets a time scale $t_L\sim L^\mu$. For $t\gg t_L$ the particles propagate across the volume and subsequent measurements of $N(t)$ at time differences $\Delta t\gg t_L$ are statistically independent, varying by an amount of order $\pm\sqrt{N}$.

At intermediate times $t\ll t_L$ the particles have a finite probability of remaining inside the volume; note that this argument depends on the existence of a mean range, i.e., $1<\mu<2$. Only particles in the boundary zone of the volume contribute to the fluctuations in $N(t)$. The thickness of the boundary zone is estimated by means of the dispersion law to be of order $l_t\sim t^{1/\mu}$. Thus, the volume of the "influence zone" is of order $l_t L^{d-1}$ and the total number of particles in the zone is $l_t L^{d-1}\rho=Nl_t/L$. The particle number in the boundary zone can be regarded to be statistically independent with fluctuations given by $\pm(Nl_t/L)^{1/2}$. Those are the only relevant fluctuations contributing to $\Delta N(t)$ for the entire volume in question and we conclude that

$$\Delta N(t)=N(t)-N(0)\sim\pm\left(\frac{Nl_t}{L}\right)^{1/2}\sim t^H \tag{3}$$

where the Hurst exponent $H=1/(2\mu)$. For the correlation function for $N(t)$ we find, correspondingly,

$$\langle\Delta N(t)^2\rangle\sim\langle[N(t)-N(0)]^2\rangle\sim\frac{N}{L}t^{2H}=\rho L^{d-1}t^{2H} \tag{4}$$

We note the explicit appearance of a factor L^{d-1} proportional to the surface of the volume.

For $0<\mu<1$ the above argument breaks down, since the elementary Lévy flight has an infinite range and can span the entire volume in a single step. As will be shown in the more detailed analysis in the next section, the Hurst exponent locks onto the value 1/2, i.e., a linear time dependence, whereas the exponent μ enters in the L-dependent prefactor, i.e.,

$$\langle \Delta N(t)^2\rangle \sim \langle [N(t)-N(0)]^2\rangle \sim \rho L^{d-\mu} t \tag{5}$$

We note that the surface dependence (for $\mu=1$) is gradually changed to a volume dependence (for $\mu=0$) as a function of $\mu=f$. It seems that this behavior must be related to the fractal dimension[3] $D=f$ of the cluster traced out by the Lévy flight.

The associated power spectrum is

$$S(\omega)=\int_0^\infty dt\,\langle [N(t)-N(0)]^2\rangle \cos(\omega t)\sim \frac{1}{\omega^{1+2H}} \tag{6}$$

The spectrum is independent of the dimension d, but depends on the Lévy distribution.

4. DENSITY CORRELATIONS

In integral form the transition probability $P(rt \mid r't')$ for independent Lévy flights is given by[3]

$$P(rt \mid r't')=\int \exp\{i\mathbf{p}(\mathbf{r}-\mathbf{r}')-p^\mu\,|t-t'|\}\,\frac{d^dp}{(2\pi)^d} \tag{7}$$

It follows that the statistically (ensemble) averaged density correlations take the form

$$\langle n(rt)\,n(r't')\rangle=\rho^2+\rho P(rt \mid r't') \tag{8}$$

where $\rho=\langle n\rangle$ is the mean density and we have used $P(rt \mid r't')=\delta^d(r-r')$. The instantaneous number of particle in a volume V is

$$N(t)=\int_V n(rt)\,d^dr \tag{9}$$

and we obtain, using Eq. (8) and inserting Eq. (7), the following expression for the particle number correlations in V:

$$\langle [N(t)-N(0)]^2\rangle=2\rho\int \frac{d^dp}{(2\pi)^d}\,[1-\exp(-p^\mu\,|t|)]\left|\int_V d^dr\,\exp(i\mathbf{pr})\right|^2 \tag{10}$$

The fluctuation spectrum is now extracted by analyzing the asymptotic properties of Eq. (10).

At short times, $1-\exp(-p^{\mu}|t|)$ rises very slowly from zero and the integral predominantly samples the large-p region. For $\mu=2$ the integral separates in Cartesian coordinates as discussed in refs. 8 and 9. In the general case the discussion is, however, most easily carried out in spherical coordinates. In the asymptotic region $p \gg 1/R$, where R is the radius of a spherical volume in d-dimensional space, we have[5]

$$\left|\int_V d^d r \exp(i\mathbf{p}\mathbf{r})\right|^2 \sim \frac{2}{\pi} R^{d-1} p^{-d-1} \cos^2\left[pR-(d+1)\frac{\pi}{4}\right] \tag{11}$$

For $R \gg 1$ the oscillating part in Eq. (11) effectively reduces the integral in Eq. (10) by a factor 1/2 and we obtain for $t \ll R^{\mu}$ the simple expression

$$\langle [N(t)-N(0)]^2\rangle \sim 2\rho \frac{S(d)}{\pi} R^{d-1} \int_{1/R}^{\infty} dp \frac{1-\exp(-p^{\mu}|t|)}{p^2} \tag{12}$$

Here $S(d)$ denotes the area of the unit sphere in d-dimensional space. We have controlled the lower limit by an effective IR cutoff $1/R$.

The above expression for the particle number correlations can be expressed in terms of the incomplete gamma function[6]; it is, however, easily discussed in the above form. For $1<\mu<2$ the integral is nonregular for small t. A simple dimensionality analysis yields the behavior $|t|^{1/\mu}$ and a more detailed analysis (see footnote 6)

$$\langle [N(t)-N(0)]^2\rangle \sim 2\rho \frac{S(d)}{\pi} \Gamma\left(1-\frac{1}{\mu}\right) R^{d-1} |t|^{1/\mu} \tag{13}$$

for $t \ll 1/R^{\mu}$ and $1<\mu<2$, where $\Gamma(1-1/\mu)$ is the gamma function.

[5] In spherical coordinates and noting that $d^d r \sim r^{d-1}\, dr \sin^{d-2}\theta\, d\theta$, where θ is a polar angle to a fixed direction, we have

$$\int_V d^d r \exp(i\mathbf{p}\mathbf{r}) \sim \int_0^R r^{d-1}\, dr \int_{-1}^{1} d\mu\, (1-u^2)^{(d-3)/2} \exp(ipr\mu)$$
$$\sim \int_0^R dr\, r^{d-1}(pr)^{1-d/2} J_{d/2-1} \sim p^{-d/2} R^{d/2} J_{d/2}(pR)$$

where $J_{d/2}$ is a Bessel function, and we obtain Eq. (11) from the asymptotic behavior

$$J_{d/2}(pR) \sim \left(\frac{2}{\pi p R}\right)^{1/2} \cos\left[pR-(d+1)\frac{\pi}{4}\right]$$

[6] In terms of the incomplete gamma function we have

$$\langle [N(t)-N(0)]^2\rangle \sim 2\rho \frac{S(d)}{\pi} R^{d-1}\left[|t|^{1/\mu}\Gamma\left(1-\frac{1}{\mu},\frac{|t|}{R^{\mu}}\right) - R\left(1-\exp\left(-\frac{|t|}{R^{\mu}}\right)\right)\right]$$

For $|t| \ll R^{\mu}$ we have $\Gamma(1-1/\mu, |t|/R^{\mu}) \to \Gamma(1-1/\mu)$ and we obtain Eq. (13).

For $0<\mu<1$ the integral changes behavior and becomes regular for small t. A simple expansion gives[7]

$$\langle [N(t)-N(0)]^2\rangle \sim 2\rho \frac{S(d)}{\pi(1-\mu)} R^{d-\mu}\,|t| \tag{14}$$

For $\mu=1$, i.e., the borderline case, we obtain a weak logarithmic correction in Eq. (14),

$$\langle [N(t)-N(0)]^2\rangle \sim -2\rho \frac{S(d)}{\pi} R^{d-1}\,|t| \log \frac{|t|}{R} \tag{15}$$

5. FIRST RETURN TIME

The power spectrum derived above can be directly connected to the distribution of time intervals T spent uninterruptedly by the Lévy walkers inside the volume V. The signal $N(t)$, equal to the total number of walkers within V at the instant t, is clearly equal to the sum over all particles of rectangular box signals $n_i(t)$: For all time instances t for which the walker number i is inside V the function $n_i(t)$ assumes the value 1. Whenever the walker is outside V the indicator function $n_i(t)$ is equal to zero. The graph of the function $n_i(t)$ consists of rectangular boxes of height 1; see Fig. 1. The duration T of the individual rectangles is governed by the distribution $D(T)$, which we are going to determine below.

Since the walkers are assumed to be independent, the power spectrum of $N(t)$ is proportional to the power spectrum of the indicator $n_i(t)$ for any

[7] The exact expression, avoiding the cutoff $1/R$, for $0<\mu<1$ is

$$\langle [N(t)-N(0)]^2\rangle = 2\rho S(d) \frac{\Gamma(1-\mu)\,\Gamma((d+\mu)/2)}{2^{1-\mu}\Gamma(1-\mu/2)^2\,\Gamma(1+(d-\mu)/2)} R^{d-\mu}\,|t|$$

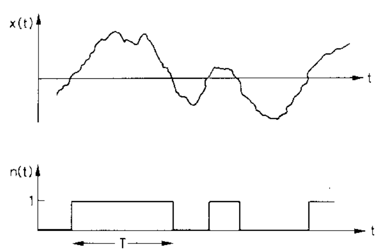

Fig. 1. The indicator function $n(t)$ for a walker in one dimension. The volume V consists of the positive real axis.

i. As shown in ref. 12, the power spectrum of $n_i(t)$ is easily found from the box size distribution $D(T)$. The scaling behavior $D(T) \propto 1/T^{\alpha}$ leads to a power spectrum $S(f) \propto 1/f^{\beta}$ with $\beta = 3 - \alpha$ when $1 \leqslant \alpha$. As α becomes less than 1, the power spectrum exponent β locks at the value 2.

We found above that the power spectrum exhibits scaling for frequencies corresponding to short times compared with the time it typically takes a walker to traverse the volume V. At this short time scale a walker entering the volume through one of the imaginary walls will leave V again in the neighborhood of where it entered. Hence, the calculation of $D(T)$ can be reduced to the problem of first return to the wall through which the walker entered V. This problem can be solved by considering instead of a bounded volume a semi-infinite region in d dimensions restricted by a hyperplane. Let V be the region for which, say, the first of the d coordinates is positive: $x_1 > 0$. The signal $n(t)$ is now given by $n(t) = \Theta[x_1(t)]$, where $\Theta[x]$ is the Heaviside step function. Assume that a walker passes into V through this hyperplane at time $T = 0$. We want to calculate the probability $D(T)$ for the walker to pass out through the hyperplane for the first time at $T > 0$.

The calculation of $D(T)$ is in fact a one-dimensional problem. We just have to keep track of the projection of the walk onto the x_1 axis. If the position $\mathbf{x}(t)$ performs a Lévy walk with index f in d dimensions, then $x_1(t)$ will execute a Lévy walk in one dimension with the *same* index f.[8]

Let us now consider a one-dimensional Lévy walk with coordinate $x(t)$. Assume that the walker in the time step from $t = -1$ to $t = 0$ makes a step from $x(-1) < 0$ to $x_0 = x(0) > 0$. The probability $P(x_0)$ for the walker in this time step to move from somewhere to the left of the origin to the position x_0 is determined by the step size distribution given in Eq. (1). We find

$$P(x_0) = \begin{cases} x_0^{-f} & \text{if} \quad x_0 > 1 \\ 1 & \text{if} \quad x_0 \in [0, 1] \end{cases} \tag{16}$$

Let $P_{x_0}(xt)$ denote the probability for the walker to be in x at time t given that it was at position x_0 at time $t = 0$. Since we are interested in the *first* time the walker passes out of the region $x > 0$ (to the region $x < 0$), it is convenient to use an absorbing boundary condition at $x = 0$.[(7)] This ensures that return to the origin is the same as first return. We use the usual image method and express $P_{x_0}(xt)$ in terms of the transition probability given in Eq. (2):

$$P_{x_0}(xt) = P(xt \mid x_0 t = 0) - P(xt \mid -x_0 t = 0) \tag{17}$$

[8] It is not strictly correct that $x_1(t)$ executes a Lévy walk. The distribution of projected displacements $D(s_1)$ scales as $1/s_1^{1+f}$ for s_1 larger than the short-distance cutoff introduced for the d-dimensional walk. For s_1 shorter than the cutoff $D(s_1)$ is a complicated function of s_1. The fact, however, that $\lim_{s_1 \to 0} D(s_1) = \text{const}$ permits us to neglect this detail.

The probability for the walker at any later time to be positioned to the right of the origin is given by

$$P_{x0}(x(t) > 0) = \int_0^\infty P_{x0}(xt)\, dx \tag{18}$$

The probability $P_{x0}(x(t) > 0)$ can only change in time due to the walker escaping through the origin. Hence, the probability $D_{x0}(T)$ for a walker that started at x_0 at time $t = 0$ to pass out through the origin at time T will be given by the time derivative of $P_{x0}(x(t) > 0)$. To obtain the first return time distribution $D(T)$, we have to average $D_{x0}(T)$ over $P(x_0)$. Collecting all the pieces, we finally have

$$D(T) = \frac{d}{dt}\int_0^\infty P(x_0)\{P(xt \mid x_0 t = 0) - P(xt \mid -x_0 t = 0)\} \tag{19}$$

where $P(x_0)$ is given in Eq. (16) and $P(xt \mid x_0 t = 0)$ in Eq. (7). It is now straightforward to extract the scaling behavior of $D(T)$ for $T \gg 1$.[9] The result is

$$D(T) \sim \frac{1}{T^{2-1/f}} \tag{20}$$

This distribution of first return times determines the distribution of box sizes in the signal $n(t) = \Theta[x(t)]$. From the scaling relation described

[9] Inserting Eq. (7) and Eq. (16) into Eq. (19) and performing the following set of substitutions in the integrals

$$p \mapsto pT^{-1/f}; \qquad x_0 \mapsto x_0 T^{1/f}; \qquad x \mapsto xT^{1/f}$$

leads to the following expression

$$D(T) = T^{-1+1/f} I_1(T) + T^{-2+1/f} I_2(T)$$

where the integrals I_1 and I_2 are defined as

$$I_1(T) = \int_0^{T^{-1/f}} dx_0 \int_0^\infty dx \int_{-\infty}^\infty \frac{dp}{2\pi} 2i \sin(px_0)\, p^f \exp\{ipx - p^f\}$$

and

$$I_2(T) = \int_{T^{-1/f}}^\infty dx_0\, x_0^{-f} \int_0^\infty dx \int_{-\infty}^\infty \frac{dp}{2\pi} 2i \sin(px_0)\, p^f \exp\{ipx - p^f\}$$

One easily finds that for $T \gg 1$

$$I_1(T) \sim T^{-2/f}; \qquad I_2(T) \sim \text{const}$$

Since $0 < f < 2$, the result in Eq. (20) follows.

above between the box size distribution and the power spectrum we find the following results:

1. $f = 2 \Rightarrow \alpha = 3/2 \Rightarrow \beta = 3/2$.
2. $1 \leqslant f < 2 \Rightarrow \alpha = 2 - 1/f \in [1, 3/2[\Rightarrow \beta = 3 - \alpha = 1 + 1/f \in]3/2, 2]$.
3. $0 \leqslant f < 1 \Rightarrow \alpha = 2 - 1/f < 1 \Rightarrow \beta = 2$.

6. SIMULATIONS

In order to illustrate the above analytical results for the fluctuation spectrum, we have performed simulations in one and two dimensions of the Lévy flight. The results are, as expected, independent of dimension.

The two-dimensional simulations are performed on a system of size $R_x \times R_y$ with periodic boundary conditions. We consider N_w independent walkers. The displacements of the walkers $\mathbf{s} = s\{\cos(\theta), \sin(\theta)\}$ are in each time step chosen stochastically. The direction is isotropic, i.e., θ is uniformly distributed on the interval $[0, 2\pi[$. The length of the displacement is distributed according to $P(s) = 1/s^{1+f}$ for $s > 1$ and $P(s) = 0$ for $s < 1$. In each time step we monitor the number of walkers $N(t)$ which, are positioned within a certain subregion V of the system.

The fluctuating signal $N(t)$ is fast-Fourier-transformed and the power spectrum calculated from the square modulus of the Fourier transform. The power spectrum is averaged over many time sequences in order to achieve sufficient statistics. The walkers wander in and out of the region V. We also measure the number of time steps T a particle spends successively within the region V. The distribution $D(T)$ of these time intervals is generated.

In Fig. 2 we show the power spectrum of 200 walkers moving on a

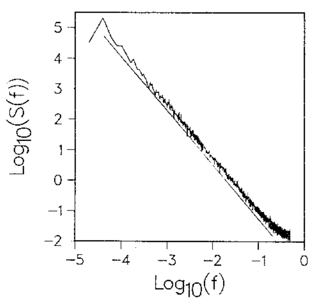

Fig. 2. Power spectrum of the fluctuations in the number of particles within a subregion V. A total of 200 Lévy walkers with index $f = 1.3$ are considered on a system with periodic boundary conditions of size $2 \cdot 10^4 \times 2 \cdot 10^4$. The region V consists of a square of linear size 10^4. The straight line has slope equal to the analytically calculated value $1 + 1/1.3 \approx 1.77$.

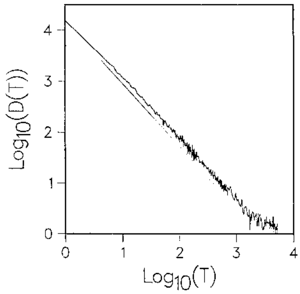

Fig. 3. Distribution of time intervals spent uninterruptedly within the subregion V. The system is the same as the one considered in Fig. 1. The straight line has slope $2-1/1.3 \approx 1.23$, which is the scaling exponent calculated for the distribution of first return times of a Lévy walker of index 1.3.

system of size $R_x = R_y = 2 \times 10^4$. The subregion V consists of a square of linear extension equal to 10^4. The Lévy flight index is $f = 1.3$. The straight line has slope equal to the analytically calculated power spectrum exponent $\beta = 1 + 1/1.3 \approx 1.77$.

In Fig. 3 we show the distribution of time intervals spent uninterruptedly within V. The straight line has the slope $\alpha = 2 - 1/1.3 \approx 1.23$ calculated above for the distribution of first return times.

There is an excellent agreement between the analytic calculation and the simulation results.

7. SUMMARY AND DISCUSSION

In this paper we have discussed in some detail the influence of an algebraic step size distribution, i.e., a Lévy distribution, for the elementary Brownian motion of particles in an ideal gas on the macroscopic particle number fluctuations in a finite test volume. The study was motivated by the desire to investigate the role of universality in this kind of dynamical system.. We find that at short times compared with the traversal time across the volume the fluctuation spectrum is characterized by a Hurst exponent which is independent of the dimension of the system.

In the case where the step size distribution falls off fast enough such that a finite mean square deviation exists, implementation of the central limit theorem implies that the Hurst exponent locks onto the universal value 1/4, characteristic of ordinary Brownian motion. We note that in the present context it is the central limit theorem which ensures universality, i.e., the decoupling between the detailed microscopic character of the Brownian motion and the macroscopic scaling behavior.

In the case where the step size distribution falls off so slowly that the mean square deviation diverges but the distribution still possesses a finite mean value, the Hurst exponent depends on the index characterizing the Lévy distribution and thus exhibits an anomalous behavior. In other words, the rare but large steps generated interfere with the macroscopic behavior of the density fluctuations and changes the scaling behavior.

Finally, in the case where the Lévy flights have infinite range, i.e., the mean step diverges, we again obtain a universal behavior as regards the time dependence of the macroscopic fluctuations in the sense that the Hurst exponent now locks onto a new value, 1/2. On the other hand, owing to the infinite range of the Lévy flights, the surface dependence of the fluctuation spectrum is changed to a fractal dependence in the sense that the fractal dimension of the cluster formed by the Lévy flights enters in the size dependence.

In a recent paper Hayot[11] has established a closure approximation in lattice gas hydrodynamics applied to turbulence by implementing microscopic Lévy walks and has derived results on the shape and flattening of velocity profiles.

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